An approximation to d' for n-alternative forced choice.

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Revised 2006 from 1990 Psychology Department publication, University of Tasmania.

An approximation suitable for computer estimation of d' from the total proportion of correct responses in an n-alternative forced choice experiment for any n is described. This approximation implies that ogival psychometric functions can be interpreted as linear in d', which provides signal detection threshold or sensitivity measures allowing comparisons between psychometric functions with different numbers of stimulus alternatives.

In signal detection studies of n-alternative forced choice (nAFC), response bias is generally small, and for 2AFC it can be shown that a moderate bias has little effect on d' (Green & Swets, 1966, p408-411). It is therefore normally assumed that response bias can be ignored, and that noise and signal plus noise distributions are normal with the same standard deviations, S_n. This allows d' to be estimated from the proportion of correct responses (Pc), by assuming that Pc is the proportion of a normal distribution with mean d' that exceeds the maximum of n-1 samples from normal distributions with mean 0, which can be obtained from equation (1) (Green & Swets,1966, p69).

$$Pc = \int_{-\infty}^{\infty} N(x-d')\left[CN(x)\right]^{n-1} dx \qquad (1)$$

where *N(x)* is the normal and *CN(x)* the cumulative normal function.

Elliott provides a table (Table II in Swets, 1964, p682-683) giving d' from Pc for selected values of n, obtained by numerical integration with limited precision of a function derived from (1) with an error less than 0.02 in Pc.

For n=2 the model estimated by (1) is the difference between two normal distributions, which is thus a normal distribution with a standard deviation $S_{ogive} = \sqrt{2}\, S_n$, (1) therefore yields a cumulative normal ogive passing through (Pc=0.5, d'=0), thus

$$d' = \frac{z(Pc) - z(0.5)}{1/\sqrt{2}} \qquad (2)$$

where *z(x)* is the inverse of the cumulative normal function.

For n>2, d'=0 implies Pc=1/n, and Elliott (in Swets, 1964, p680) notes that the function relating Pc to d' is closely approximated by a cumulative normal ogive passing through that point, generated from a normal distribution with standard deviation $S_{ogive} = S_n/An$. Thus d' can be approximated by

$$d' = \frac{z(Pc) - z(1/n)}{An} \qquad (3)$$

where $A_2 = 1/\sqrt{2}$ and An tends towards 1 as n increases.

Elliott obtained empirical estimates for An from normal ogive approximations to the tables derived from (1) for a substantial range of n, which are presented in a graph (Fig 5 in Swets, 1964, p681) with relatively poor resolution. Green & Birdsall provide a table of An for selected values of n (table A2 in Swets, 1964, p617).

To provide a convenient computer program for estimating d' from Pc using the normal ogive approximation, values of An were estimated by numerically integrating (1) for selected values of d' and n and substituting in (3), examples of these appear in Table 1.

Table 1. Computed Percentage of Correct Responses (%c =100xPc) and the Corresponding Estimates of An.

	n=2		n=3		n=4		n=8		n=16	
d'	%c	An	%c	An	%c	An	%c	An	%c	An
-1	24.0	.707	11.3	.780	6.8	.815	2.2	.870	0.7	.904
0	50.0	-	33.3	-	25.0	-	12.5	-	6.2	-
1	76.0	.707	63.4	.771	55.2	.805	38.5	.860	26.1	.893
2	92.1	.707	86.6	.768	82.3	.800	71.1	.853	59.5	.887
3	98.3	.707	96.9	.765	95.6	.795	91.8	.847	86.6	.881

For n>2 An varies with d' because of the error in the cumulative normal approximation to (1). The value of An at 75% correct was selected for use in approximating d' as this provides reasonable approximations over the useful performance range (above chance to about 95% correct). It was found that 1/(1-An) was closely approximated by a quadratic in log(n), giving (4) in log to base 10.

$$An \approx 1 - \frac{1}{1.93 + 4.75\log(n) + 0.63[\log(n)]^2} \qquad (4)$$

Table 2 compares the values of An for 75% correct obtained by integration with those obtained from (4) and those in Green & Birdsall's table A2 (Swets, 1964, p617).

Table 2. Estimates of An.

n	75%c by Integration	Estimated from (4)	Green and Birdsall
2	.707	.707	.707
3	.770	.770	__
4	.801	.801	.827
8	.852	.852	.855
16	.884	.883	.884
32	.905	.905	.890
64	.921	.920	__
256	.940	.941	.916
1000	.952	.954	.964
10000	.964	.968	__
100000	.971	.976	__

Green & Birdsall do not explain how their Table A2 values were derived so it is not possible to determine how the discrepancies between them and these estimates occurred, but these estimates are consistent with the limited precision of individual entries in Elliott's Table II. Elliott's Fig 5 (Swets, 1964, p 681) shows A4 clearly less than 0.827, and A32 and A256 appear closer to the values obtained here than to those in Green & Birdsall's table.

Using the values of An from (4) in (3) gives values of d' with an error of less than 2% (mostly less than 1%) from those obtained by integration for the range d'=0 (or 1% correct for n>1000) to 75% correct and an error of less than 4% up to 95% correct for n up to at least 10000, and slightly greater maximum errors for n=100000. This approximation is comparable to the accuracy of Elliott's table (0.02 in proportion correct) but can be used for any n.

Applications

Appendix A contains Pascal functions An, for estimating An, and DfromPc, for estimating d' from Pc. DfromPc requires a function ZfromP for the inverse of the normal ogive, the version in Appendix A is based on Hastings (1955, p192) and offers 3 decimal place accuracy, which is more than required for d' estimation but is more generally useful.

It is essential to prevent calls to DfromPc with Pc=1 (which is likely) and Pc=0. The common practice is to set the minimum number correct and number of errors to 0.5 in calculating Pc, or to define

$$Pc' = (n_{correct} + 0.5)/(n_{correct} + n_{incorrect} + 1.0) \qquad (5)$$

If the objective is to fit a straight line in d', as in a possible signal detection model of psychometric functions, it is preferable to fit a normal ogive to the Pc, to reduce the weight of Pc's near 1.0 (which give large and unreliable d'). With ogive fitting it is not necessary to adjust individual Pc as in (5), but it is necessary to protect against Pc=1 for all stimulus values, and against fits below chance performance. If it can be assumed that the straight line in d' should pass through the origin, the normal ogives fitted should be constrained to pass through Pc=1/n at stimulus value=0, minimizing the sum of squared deviations in Pc for the parameter S_{ogive}. Performance can then be characterised as sensitivity (6), the slope of d' against stimulus value, or as the d'=1 threshold (7), equivalent to the 76% correct threshold for 2AFC.

$$\text{sensitivity} = 1/(An\ S_{ogive}) \qquad (6)$$

$$\text{threshold} = An\ S_{ogive} \qquad (7)$$

Appendix B refers to implementation in Excel.

Example of an application

Smith (1986) obtained Pc in a 2, 4 and 8 light choice reaction time experiment, where the stimulus was masked by all 8 lights after a range of stimulus exposure durations. When exposure was not limited, Pc averaged about 0.9 for 4 and 8 lights, possibly because of the emphasis on speed of responding in this experiment. Using this Pc as asymptotic performance, Smith (1986) found for each n the Pc were well fitted by a normal ogive, with S_{ogive} about 35ms for all 3 curves. Table 3 shows the S_{ogive} values obtained by Smith(1986), and the estimated sensitivity and threshold measures proposed in (6) and (7).

Table 3. Performance Parameters for Smith's (1986) Experiment

n	S_{ogive} ms	Sensitivity /ms	Threshold ms
2	34.9	.040	24.7
4	32.4	.039	26.0
8	35.1	.033	29.9

While Smith (1986) argues that the apparent constancy of S_{ogive} suggests that noise in the decision process may be attached to the decision rather than to the number of stimuli, the estimated signal detection thresholds do not seem inconsistent with a signal detection interpretation, that under the conditions in this experiment, 8 lights are somewhat less discriminable than 4 or 2, possibly because of the greater angle they subtended. Smith (1986) assumes that if noise were an attribute of the individual stimuli, as in signal detection theory, then S_{ogive} should increase with the number of alternatives, but performance in the signal detection model depends on the maximum of the n-1 noise stimuli rather than their sum. While the mean of this maximum increases with n, its standard deviation, which affects S_{ogive}, decreases with n. Thus in the signal detection model if signal strength is constant, S_{ogive} should decrease with n, proportionally to 1/An which decreases relatively slowly for n>2. Reversing the sign of the signal detection prediction does not invalidate Smith's (1986) argument, which is based on the apparent constancy of S_{ogive}, but it is necessary to establish whether this finding is reliable enough to discriminate constant noise from signal detection models.

References

Green, D. M. & Swets, J. A. (1966). *Signal detection theory and psychophysics*. New York: Wiley.

Hastings, C. (1955). *Approximations for digital computers*. Princeton: Princeton University Press.

Smith, G. A. (1986). Inspection time and response strategies in a choice response task. *Personality and individual differences*, **7**, 701-707.

Swets, J. A. (1964). *Signal detection and recognition by human observers*. New York: Wiley.

Appendix A: Pascal functions

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 (* log is base 10, ln is log to base e, sqr is square, sqrt is square root, a decimal followed
    by ei means multiplied by 10^i *)
Function ZfromP(P:real) :real;
 (* inverse of cumulative normal, returns -999 for P<1.0e-12, 999  for P>1 - 1.0e-12, 3
    decimal place accuracy otherwise *)
var T1, T2, Z :real; Zsign :integer;
begin
    if P > 0.5 then
      begin
      P :=1 - P; Zsign :=1;
      end
    else Zsign :=-1;
    if P < 1.0e-12 then Z :=999
      else
      begin
      T1 := -2*ln(P); T2 := sqrt(T1);
      Z := T2 - (2.515517 + 0.802853*T2 + 1.0328e-2*T1)/
            (1.0 + 1.432788*T2 + 1.308e-3*T1*T2);
      end; (* else *)
    ZfromP := Zsign*Z;
end; (* Zfrom P *)

Function An(N:integer) :real;
(* provides An for the normal approximation to dprime for nAFC*)
begin
    An:= 1 - 1/(1.93 + 4.75*log(N) + 0.63*sqr(log(N)));
end; (* An *)

Function DfromPc(Pc:real; N:integer) :real;
 (* calculates dprime by the normal approximation for nAFC given proportion correct,
    0<Pc<1, and the number of choices, N>1 *)
begin
    DfromPc := (ZfromP(Pc) - ZfromP(1/N))/An(N);
end; (* DfromPc *)
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Appendix B: Excel

In Excel Log10 is log to base 10, Normsinv returns z from p for a cumulative normal function.

The file nAFC.xls is an Excel file: the first sheet combines (3) and (4) to estimate d' from n and Pc.

If cell B2 contains proportion correct and cell A2 contains n (the number of choices) then d' is estimated by:

=(NORMSINV(B2)-NORMSINV(1/A2))/(1-1/(1.93+4.75*LOG10(A2)+0.63*LOG10(A2)*LOG10(A2)))

The second sheet implements (4) to estimate An from n, if n is in cell a2, An is estimated by:

=(1-1/(1.93+4.75*LOG10(A2)+0.63*LOG10(A2)*LOG10(A2)))